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Collocation Approximation Techniques for Solving Integro-Differential Equations Via Shifted Chebyshev Polynomials

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Abstract

In this work, an approximate solution of linear second kind integro-differential equations of the Volterra and Fredholm types was studied utilizing shifted Chebyshev polynomials as basic functions. Additionally, the approximate solution was collocated using standard collocation and Chebyshev Gauss Lobatto collocation points, respectively. In terms of obtained errors, comparisons were done with the two collocation points. The performance of the method was demonstrated numerically in terms of the degree of approximation. Nonetheless, it was found that, Chebyshev Gauss-Lobatto collocation points exhibit better accuracy than standard collocation points, as can be seen from the tables of errors presented.

Keywords: Approximate solution, Chebyshev Gauss-Lobatto, Shifted Chebyshev polynomials, Standard collocation, integro-differential equations.

1. Introduction

Integro-differential equations represent a category of equations that encompass both differential and integral operators. These equations emerge in diverse fields of study such as physics, biology, and finance. The act of analytically solving these equations often presents considerable challenges, leading to the prevalent utilization of numerical methods. Among the various numerical techniques at our disposal, the implementation of collocation approximation has demonstrated its effectiveness in addressing integro-differential equations (Dung, 2021). Within the scope of this project research, our emphasis lies in the application of collocation approximation techniques for the resolution of integro-differential equations through the utilization of shifted Chebyshev polynomials. Collocation approximation involves choosing a set of collocation points in the domain of the problem and approximating the solution by a polynomial that satisfies the equation at these collocation points. The accuracy of the approximation depends on the choice of collocation points and the type of basic functions used. Chebyshev polynomials, particularly shifted Chebyshev polynomials, have been widely used as basic functions in collocation methods due to their excellent approximation properties (Anselmann et al., 2019).

The utilization of collocation approximation techniques for the resolution of integro-differential equations has garnered considerable attention in recent years. Scholars have devised diverse collocation methodologies reliant on distinct categories of basic functions, such as Jacobi polynomials (Bhrawy et al., 2016), tau-collocation (Mamadu & Njoseh, 2016), and shifted Chebyshev polynomials. These methodologies have been effectively employed in tackling integro-differential equations emerging in various domains, thereby showcasing their efficacy and versatility.

The use of collocation approximation techniques for solving integro-differential equations has gained significant attention in recent years. Researchers have developed various collocation methods based on different types of basis functions, such as Jacobi polynomials (Bhrawy et al., 2016), tau-collocation (Mamadu & Njoseh, 2016), and shifted Chebyshev polynomials. These methods have been successfully applied to solve integro-differential equations arising in different domains, demonstrating their effectiveness and versatility.

The great work did by the researchers aforementioned motivated us and eventually led to the proposal of a numerical approximation method that is efficient and accurate with less computational work to obtain an approximate solution of linear second kind volterra and fredholm integro-differential equations of the form:

$$b_j\xi'' + b_j\xi' + b_j\xi(x) + \lambda\int_{c(x)}^{e(x)} Q(x,\ t)\,\xi(t)\,dt = f(x) \tag{1}$$

subject to the initial conditions;

$$\xi(0) = a\ ,\ \xi'(0) = b \tag{2}$$

where c $and\,e$ are finite constants.$f(x)$ and $Q(x,\ t)$ are given real-valued functions and $\xi's$ are unknown constants to be determined.

This paper aimed to explore and evaluate the effectiveness of collocation approximation techniques for solving linear secod kind of Volterra and Fredholm integro-differential equations using shifted Chebyshev polynomials. Thus,the main objectives are to transform the integro-differential equation in equation (1) subject to initial conditions in equation (2) into a system of linear algebraic equations, test the efficiency of the collocation points on some numerical examples, and compare the two collocation points with each other.

2. BASIC DEFINITIONS

Definition 2.1:

Integro-Differential Equation (Wazwaz 2011)

Integro-differential equations are mathematical equations that involve both derivatives and integrals. They express relationships between a function, its derivatives, and its integrals. These equations frequently arise in many scientific fields, including physics, biology, and engineering. Solving integro-differential equations is challenging due to their inherent complexity and nonlinearity, often requiring advanced mathematical techniques such as collocation approximation.

Definition 2.2:

Exact Solution (Tescan, 2009)

A solution is called exact solution if it can be expressed in a closed-form such as a polynomial, exponential function, trigonometric function, or the combination of two or more of these elementary functions.

Definition 2.3:

Approximate Solution (Sarafyan, 1994)

An approximate solution is an inexact representation of the exact solution that is still close enough to be useful, in this work, approximate solutions used is given by:

$$\xi(x) = \sum_{j=o}^{M} d_j P_j(x) \tag{2.1}$$

where $dj(j \geq 0)$ are the unknown constants to be determined and $P_J(x)$ are the approximate solution.

Definition 2.4:

Chebyshev polynomials (Rahman, Hakim, Hasan, Alam & Ali, 2012)

The general form of the Chebyshev polynomials of mth degree is defined by

$$\xi_m(x) = \sum_{i=0}^{\frac{m}{2}} (-1)^i \frac{m!}{(2i)!(m-2i)!} (1-x^2)^i x^{m-2i} \qquad (2.2)$$

using Maple code, the first few Chebyshev polynomials over the interval of [-1, 1] are given as

$$\left.\begin{array}{r} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = 2x^2 - 1 \\ P_3(x) = 4x^3 - 3x \\ P_4(x) = 8x^4 - 8x^2 + 1 \end{array}\right\} \qquad (2.3)$$

Definition 2.5:

Conversion of Chebyshev polynomials (Adebisi, Taiwo, Adewumi & Bello, 2019)

The Chebyshev polynomial denoted by $P_i(x)$ and valid in the interval $a \le x \le b$ is defined as

$$P_i(x) = \cos\left[icos^{-1}\left(\frac{2x-(a+b)}{b-a}\right)\right] \quad , \; i = 0,1,2,\ldots \qquad (2.4)$$

and the recurrence relation is given as

$$P_{i+1}(x) = 2\left[\frac{2x-(a-b)}{b-a}\right]P_i(x) - P_{i-1}(x), \qquad a \le x \le b \qquad (2.5)$$

To change from interval [a, b] to [0, 1] the following procedure is carefully followed. Here, we put $a = 0$ and $b = 1$ in equations (2.4), we obtain

$$P_i(x) = \cos\left[icos^{-1}\left(\frac{2x-1}{1-0}\right)\right] \; , \qquad (2.6)$$

$$P_i(x) = \cos(\,icos^{-1}(2x-1)) \qquad (2.7)$$

For the purpose of this work, we need to transform $P_i(x)$ in [-1, 1] to $P_i(x)$ in [0, 1]

If $x \,\epsilon\, [-1,1] \rightarrow v \,\epsilon\, [0,1]$

Let $v = \gamma z + \lambda$, where γ and λ are some parameters. Therefore

$$0 = -\gamma + \lambda \tag{2.8}$$

$$1 = \gamma + \lambda \tag{2.9}$$

Solving equations (2.8) and (2.9) simultaneously, we have

$$2\gamma = 1 \Rightarrow \gamma \tag{2.10}$$

$$\Rightarrow \lambda = \gamma = \frac{1}{2} \tag{2.11}$$

$$v = \gamma z + \lambda = \frac{1}{2}\mathrm{x} + \frac{1}{2} = \frac{x+1}{2} \tag{2.12}$$

By substituting the values of γ and λ, we have $x = 2v - 1$

Thus, we have the shifted Chebyshev polynomials

$$P_i^*(z) = P_i^*(2v-1) \tag{2.13}$$

Therefore,

$$P_i(x) = \cos(\,icos^{-1}(2v-1)), \qquad for \quad 0 \le v \le 1\ , \tag{2.14}$$

Since v is a dummy variable

$$P_i^*(x) = \cos(\,icos^{-1}(2x-1)), \qquad for \quad 0 \le v \le 1\ , \Rightarrow v = x \tag{2.15}$$

Since

$$P_{i+1}(z) = 2xP_i(z) - P_{i-1}(x), \qquad for\ -1 \le x \le 1 \tag{2.16}$$

then

$$P_{i+1}^*(x) = 2(2v-1)P_i^*(z) - P_{i-1}^*(x), \qquad for\ 0 \le x \le 1 \tag{2.17}$$

Hence, shifted Chebyshev polynomial formula

when

$$\left.\begin{array}{ll} i = 0 : P_0^*(x) = & 1 \\ i = 1 : P_1^*(x) = & 2x - 1 \\ i = 2 : P_2^*(x) = & 8x^2 - 8x + 1 \\ i = 3 : P_3^*(x) = & 32x^2 - 48x^2 + 18x - 1 \\ i = 4 : P_4^*(x) = & 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ i = 5 : P_5^*(x) = & 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \end{array}\right\} \tag{2.18}$$

and so on.

Hence, the trial solution is now given as

$$\xi(x) = \sum_{i=0}^{M} d_i P_i^*(x) \tag{2.19}$$

where d_i $(i \geq 0)$ are the unknown constants to be determined and $P_i^*(x)$ is shifted Chebyshev polynomials of any kind and M is the degree of approximating polynomials, where in most cases the better approximate solution(i.e. closer to exact solution) is produced by the larger M and d_i is the specialized coordinate called degree of freedom.

3.0. Problem considered and Methodology

In this section, we employed the general integro-differential equation of the form;

$$b_j\xi'' + b_j\xi' + b_j\xi(x) + \lambda \int_{c(x)}^{e(x)} Q(x,\ t)\ \xi(t)\ dt = f(x) \tag{3.1}$$

subject to the initial conditions;

$$\xi(0) = a, \xi'(0) = b \tag{3.2}$$

where c(x) and $e(x)$ are the limits of integration which may be constants, variables or combined, ξ(t) is a constant parameter, f(x) is a given function and $Q(x,\ t)$ is called kernel of function.

3.1 Methodology

3.1.1 Standard collocation points by shifted Chebyshev polynomials

In order to solve equations (3.1) and (3.2) using standard collocation method via Chebyshev polynomial, we set an approximate solution of the form

$$\xi_n(x) = \sum_{j=0}^{N} d_j P_j^*(x) \tag{3.3}$$

Where N is the degree of our approximation and d_j represents the unknown constants to be determined and $P_j^*(x)$ is the shifted Chebyshev polynomials defined in equation (1.18).

Thus differentiating equation (3.3) twice, we obtain,

$$\left.\begin{aligned} \xi' &= \sum_{j=0}^{N} d_j P_j^{*\prime}(x) \\ \xi'' &= \sum_{j=0}^{N} d_j P_j^{*\prime\prime}(x) \end{aligned}\right\} \tag{3.4}$$

Thus, putting equation (3.3) into equation (3.4) we have;

$$b_j \sum_{j=0}^{N} d_j P_j^{*\prime\prime}(x) + b_j \sum_{j=0}^{N} d_j P_j^{*\prime}(x) + b_j \sum_{j=0}^{N} d_j P_j^{*}(x) + \lambda \int_{c(x)}^{e(x)} Q(x,t) \left(\sum_{i=0}^{N} d_j P_j^{*}(t) \right) dt = f \tag{3.5}$$

Evaluating the integral in equation (3.5) above, we have;

$$b_j \sum_{j=0}^{N} d_j P_j^{*\prime\prime}(x) + b_j \sum_{j=0}^{N} d_j P_j^{*\prime}(x) + b_j \sum_{j=0}^{N} d_j P_j^{*}(x) + \lambda \mathrm{H(x)} = f(x) \tag{3.6}$$

Where, $\mathrm{H(x)} = \lambda \int_{c(x)}^{e(x)} Q(x,\ t) \left(\sum_{i=0}^{N} d_j P_j^{*}(t) \right) dt$

Hence, collocation at point x $= x_n$ we have

$$b_j \sum_{j=0}^{N} d_j P_j^{*\prime\prime}(x_n) + b_j \sum_{j=0}^{N} d_j P_j^{*\prime}(x_n) + b_j \sum_{j=0}^{N} d_j P_j^{*}(x_n) + \lambda \mathrm{H(x)} = f(x) \tag{3.7}$$

where $x_n = a + \frac{(b-a)n}{N}$; n= 1, 2, … , N – 1 (3.8)

Thus, equation (3.7) is then put into matrix form as

$$Pd = q(x) \tag{3.9}$$

Where,

$$p = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & \cdots & p_{1n} \\ \mathrm{p}_{21} & p_{22} & p_{23} & p_{24} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {p_{11}}^{0} & {p_{12}}^{0} & {p_{13}}^{0} & {p_{14}}^{0} & \cdots & {p_{1n}}^{0} \\ {p_{21}}^{0} & {p_{22}}^{0} & {p_{11}}^{0} & {p_{11}}^{0} & \cdots & {p_{21}}^{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {p_{m1}}^{n-1} & {p_{m2}}^{n-1} & {p_{m3}}^{n-1} & {p_{m4}}^{n-1} & \cdots & {p_{mn}}^{n-1} \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ d_M \end{pmatrix} = \begin{pmatrix} q_{11} \\ q_{21} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ {q_{mn}}^{n-1} \end{pmatrix}, \tag{3.10}$$

Equation (3.7) produces (N– 1) algebraic equations in the (N + 1) unknown constants. Two extra equations are formed from the conditions in equation (3.2). Altogether, we now have (N+1) algebraic equations in (N + 1) unknown constants. These equations are then solved using Maple 18 to have (N+ 1) unknown constants d_j(j ≥ 0) which then put back into the approximate solution given by equation (3.4).

3.1.2 Chebyshev Gauss-Lobatto collocation points shifted Chebyshev polynomials defined by

In order to discuss this method, we assumed the collocation points of the form

$$x_n = \cos\left(\frac{\pi n}{N}\right),\ n = 1, 2, \cdots, N \tag{3.11}$$

This method is similar to the one we discussed above under section (3.1.2). We have (N+1) algebraic linear system of equation (N+1) unknown constants d_j $(d_j \geq 0)$ The (N+1) linear algebraic system of equation is then solved by maple 18 software to obtain the unknown constants d_j $(d_j \geq 0)$ which are substituted back into equation (3.4)

4. Numerical Examples and Results

In this section, we have demonstrated the standard collocation and Chebyshev Gauss Lobatto collocation approximation method on the second kind integro-differential equations using Chebyshev polynomial as the basis function. The results obtained are compared with each other on three problems to test for the effectiveness and efficiency of our collocation points via the Maple 18 software.

Numerical Example 1

Consider the integro-differential equation of the form (Olayiwola *et al.,* 2020)

$$\boldsymbol{\xi''(x) = -x - \frac{x^3}{6} + \int_0^x (x - t)}\xi(t)dt \tag{4.1}$$

Subject to the initial value conditions

$$\xi(0) = 0, \xi'(0) = 2 \tag{4.2}$$

Also, given the exact solution to be;

$$\xi(x) = x + \sin x \tag{4.3}$$

Numerical Example 2

Consider the integro-differential equation of the form (Wazwaz, 2011)

$$\xi''(x) = e^x - \frac{4}{3}x + x\int_0^1 t\xi(t)dt \tag{4.4}$$

Subject to the initial conditions

$$\xi(0) = 0,\ \xi'(0) = 2 \tag{4.5}$$

Also, given the exact solution to be;

$$\xi(x) = x + e^{x} \tag{4.6}$$

Numerical Example 3

Consider the integro-differential equation of the form **(Wazwaz, 2015)**

$$\xi''(x) = 2x - cosx + \int_0^{\pi} xt\xi(t)dt \tag{4.7}$$

Subject to the initial conditions

$$\xi(0) = 0,\ \xi'(0) = 1 \tag{4.8}$$

Also, given the exact solution to be;

$$\xi(x) = \cos x \tag{4.9}$$

Remark:

Here, we present the results obtained by solving three examples by the methods discussed in chapter four using Shifted Chebyshev polynomial as our basis functions. The following acronyms are adopted in the tables below

SCP: Standard Collocation Points;

CGLCP: Chebyshev Gauss-Lobatto Collocation Points

We define our Absolute error as;

$$Error = |\xi(Exact) - \xi(Approximate)|, c \leq x \leq e$$

Table 1. Results and Absolute errors obtained for example 1 Using SCP for case M = 2

x	$\xi(Exact)$	$CGLCP(Error)$	$SCP(error)$
0.0	0.000000000	$\mathbf{4.0000} \times \mathbf{10^{-11}}$	$5.00000 \times \mathbf{10^{-11}}$
0.2	0.3986693308	$\mathbf{5.6259} \times \mathbf{10^{-3}}$	$\mathbf{5.2159} \times \mathbf{10^{-3}}$
0.4	0.78941834231	$1.7244 \times \mathbf{10^{-2}}$	$1.7826 \times \mathbf{10^{-2}}$
0.6	**1.1646424734**	$2.7251 \times \mathbf{10^{-2}}$	$3.5617 \times \mathbf{10^{-3}}$
0.8	**1.5173560909**	$2.8660 \times \mathbf{10^{-2}}$	$2.2101 \times \mathbf{10^{-2}}$
1.0	**1.841470985**	$1.5471 \times \mathbf{10^{-2}}$	$6.3650 \times \mathbf{10^{-2}}$

Table 2. Results and Absolute errors obtained for example 2 Using SCP for case M = 3

x	$\xi(Exact)$	$CGLCP(Error)$	$SCP(error)$
0.0	1. 000000000	3.0000×10^{-10}	$0.00000$0000
0.2	1.4214027582	9.1199×10^{-5}	7.6673×10^{-3}
0.4	1.8918246976	3.0009×10^{-3}	3.2561×10^{-3}
0.6	**2.422118800**	4.1717×10^{-2}	8.8520×10^{-4}
0.8	**3.025540928**	5.9699×10^{-2}	4.8119×10^{-3}
1.0	**3.718281828**	7.6141×10^{-2}	1.5132×10^{-3}

Table 3. Results and Absolute errors obtained for example 3 Using SCP for case M = 4

x	$\xi(Exact)$	$CGLCP(Error)$	$SCP(error)$
0.0	1.00000000	0.00000000	0.000000000
0.2	0.98006658	4.5062×10^{-3}	4.9020×10^{-4}
0.4	0.92106099	7.1845×10^{-4}	2.1770×10^{-5}
0.6	**0.8253356**	3.8438×10^{-3}	7.5330×10^{-4}
0.8	**0.6967067**	1.3045×10^{-3}	2.0550×10^{-4}
1.0	**0.5403023**	8.5213×10^{-3}	5.9771×10^{-3}

Table 1, 2, and 3, show the numerical solution obtained in terms of approximate solution and the errors for the linear second kind integro-differential equations of Volterra and Fredholm types solved through standard collocation and Chebyshev Gauss Lobatto collocation points using shifted Chebyshev polynomial as basis function. We also observed from the examples solved that both collocation points converge close enough to the exact solution in a view iteration and lower error.

Conclusion

In this work, collocation approximation techniques for solving integro-differential equations via shifted Chebyshev polynomial was employed and observed, however, as M increases, the results obtained yield a good approximation of the exact solution only in view iterations for all the problems considered (as can be seen in the tables of errors). Also, using Standard and Chebyshev Gauss Lobatto collocation point's exhibit straightforward and accurate results. In this work, Standard collocation points provide more accurate results than Chebyshev Gauss Lobatto Collocation points. Furthermore, the entire problem was solved using the MAPLE 18 software.

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